Critical Casimir forces for O(n) systems with long-range interaction in the spherical limit

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(Received 25 June 2004; published 2 December 2004)

We present exact results on the behavior of the thermodynamic Casimir force and the excess free energy in the framework of the d-dimensional spherical model with a power law long-ranged interaction decaying at large distances r as $r^{-d-\sigma}$, where $\sigma < d < 2\sigma$ and $0 < \sigma \leqslant 2$. For a film geometry and under periodic boundary conditions we consider the behavior of these quantities near the bulk critical temperature T_c, as well as for $T > T_c$ and $T < T_c$. The universal finite-size scaling function governing the behavior of the force in the critical region is derived and its asymptotics are investigated. While in the critical and subcritical region the force is of the order of L^{-d}, for $T > T_c$ it decays as $L^{-d-\sigma}$, where L is the thickness of the film. We consider both the case of a finite system that has no phase transition of its own, when $d-1 < \sigma$, as well as the case with $d-1 > \sigma$, when one observes a dimensional crossover from d to a $d-1$ dimensional critical behavior. The behavior of the force along the phase coexistence line for a magnetic field $H=0$ and $T < T_c$ is also derived. We have proven analytically that the excess free energy is always negative and monotonically increasing function of T and H. For the Casimir force we have demonstrated that for any $\sigma \geqslant 1$ it is everywhere negative, i.e., an attraction between the surfaces bounding the system is to be observed. At $T=T_c$ the force is an increasing function of T for $\sigma > 1$ and a decreasing one for $\sigma < 1$. For any d and σ the minimum of the force at $T=T_c$ is always achieved at some $H \neq 0$.

DOI: 10.1103/PhysRevE.70.066106 PACS number(s): 05.70.Jk, 64.60.Fr, 68.15.+e, 68.35.Rh

I. INTRODUCTION

When a fluid is confined in a film geometry with a thickness L, the boundary conditions which the order parameter has to fulfill at the surfaces bounding the system lead to a L dependence of the excess free energy. On its turn, the last lead to a force, conjugated to L, which is called the Casimir (solvation) force and the corresponding effect—the thermodynamic Casimir effect. In this form it has been discussed for the first time by Fisher and de Gennes in 1978 [1]. The effect is dubbed so after the Dutch physicist Hendrik Casimir who first, in 1948 [2], predicted it considering the influence of the zero-point quantum mechanical vacuum fluctuations of the electromagnetic field on the resulting force between two infinite perfectly conducting planes placed against each other. In that form the effect is known as the quantum mechanical Casimir effect. For a long time the effect was considered as a theoretical curiosity but the interest in it has blossomed in the past decade. Numerous calculations and experiments have been performed both on the thermodynamic and the quantum Casimir effect. For a review on the thermodynamic effect the interested reader might consult [3–5], and for the quantum one [6–9].

The Casimir force in statistical-mechanical systems at a temperature T and in the presence of an external magnetic field H is characterized by the excess free energy due to the *finite-size contributions* to the total free energy of the system. In the case of a film geometry $L \times \infty^2$, and under given boundary conditions τ imposed across the direction L, the Casimir force is defined as

$$F^{\tau}_{\rm Casimir}(T,H,L) = -\frac{\partial f^{\rm ex}_{\tau}(T,H,L)}{\partial L}, \tag{1.1}$$

where $f^{\rm ex}_{\tau}(T,H,L)$ is the excess free energy

$$f^{\rm ex}_{\tau}(T,H,L) = f_{\tau}(T,H,L) - L f_{\rm bulk}(T,H). \tag{1.2}$$

Here $f_{\tau}(T,H,L)$ is the full free energy per unit area and per $k_B T$, and $f_{\rm bulk}(T,H)$ is the corresponding bulk free energy density. According to the standard finite-size scaling theory [5,10], under periodic boundary conditions $\tau=p$ near the critical point $T=T_c, H=0$ (of the bulk system) one expects

$$f^{\rm ex}_{p}(T,H,L) = L^{-(d-1)} X^{(p)}_{f}(atL^{1/\nu}, bhL^{\Delta/\nu}), \tag{1.3}$$

wherefrom one has

$$F^{(p)}_{\rm Casimir}(T,H,L) = L^{-d} X^{(p)}_{\rm Casimir}(atL^{1/\nu}, bhL^{\Delta/\nu}). \tag{1.4}$$

Here the universal scaling functions of the free energy $X^{(p)}_{f}(x_1,x_2)$ and the Casimir force $X^{(p)}_{\rm Casimr}(x_1,x_2)$ are related via the relation

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$$X_{\text{Casimir}}^{(p)}(x_1,x_2) = (d-1)X_f^{(p)}(x_1,x_2) - \frac{1}{\nu}x_1\frac{\partial}{\partial x_1}X_f^{(p)}(x_1,x_2) - \frac{\Delta}{\nu}x_2\frac{\partial}{\partial x_2}X_f^{(p)}(x_1,x_2), \quad (1.5)$$

Δ and ν are the standard critical exponents, a and b are nonuniversal metric factors, $t=(T-T_c)/T_c$ is the reduced temperature and $h=\beta H$, with $\beta=(k_BT)^{-1}$. We recall that, according to the general theory of the thermodynamic Casimir effect [3–5], $X_{\text{Casimir}}^{(p)}(x_1,x_2)$ is supposed to be negative under periodic boundary conditions (which corresponds to a mutual attraction of the "surfaces" bounding the system). The boundaries influence the system to a depth given by the bulk correlation length $\xi_\infty(T)\sim|T-T_c|^{-\nu}$, where ν is its critical exponent. When $\xi_\infty(T)\ll L$ the Casimir force, as a *fluctuation induced force* between the plates, is negligible. The force becomes long-ranged when $\xi_\infty(T)$ diverges near *and* below the bulk critical point T_c in an $\mathrm{O}(n), n\geq 2$ model system in the absence of an external magnetic field [11–13]. Therefore in statistical-mechanical systems one can turn on and off the Casimir effect merely by changing, e.g., the temperature of the system.

The temperature dependence of the Casimir force for two-dimensional systems has been investigated exactly only on the example of Ising strips [14]. In $\mathrm{O}(n)$ models for $T>T_c$ the temperature dependence of the force has been considered in [11]. The only example where it is investigated exactly as a function of both the temperature and of the magnetic field scaling variables is that of the three-dimensional spherical model with short range interaction under periodic boundary conditions [12,13,15]. There results for the Casimir force in a mean-spherical model with $L\times\infty^{d-1}$ geometry, $2<d<4$, have been derived. The force is consistent with an *attraction* of the plates confining the system. In [16] some of the results of [12,13] have been extended to a quantum version of the model. There the interaction has been taken to be long-ranged, with $0<\sigma\leq 2$, where $\sigma/2<d<3\sigma/2$, and the corresponding quantum phase transition has been considered to take place at $T=0$. Very recently in [15], based on a derived there stress-tensor-like operator for critical lattice systems, the scaling functions of the force for the 3D Ising, *XY* and Heisenberg models have been obtained by Monte Carlo methods. The results suggest that, under periodic boundary conditions, the scaling function $X_{\text{Casimir}}^{(p)}(x)/n$ of all $\mathrm{O}(n)$ models practically coincide for large x, say, for $x=L/\xi\gtrsim 2$, where ξ is the true bulk correlation length. The last increases the helpfulness of the spherical model results (i.e., of the results corresponding to the limit $n\to\infty$), which are available in an explicit analytic form.

Most of the results for the Casimir force are available only at $T=T_c$, i.e., for the Casimir amplitudes. They are obtained for $d=2$ by using conformal-invariance methods for a large class of models [3]. For $d\neq 2$ results for the amplitudes are available via field-theoretical renormalization group theory in $4-\varepsilon$ dimensions [3,11,17], Migdal-Kadanoff real-space renormalization group methods [18], and by Monte Carlo methods [15,19]. In addition to the flat geometries some results about the Casimir amplitudes between spherical particles in a critical fluid have been derived too [17,20]. For the purposes of experimental verification that type of geometry seems especially suitable. For $d=3$ the only exactly known amplitude is that one for the spherical model [13]. In the case $d=\sigma$ the amplitude is also known [16] for the quantum version of the model with long-ranged power-law interaction (in that case the amplitude in question characterizes the leading temperature correction to the ground state of the quantum system).

It should be noted that in contrast to the quantum mechanical Casimir effect, that has been tested experimentally with high accuracy [21–24] (for a recent review on the existing experiments see, e.g. [25]), the statistical-mechanical Casimir effect lacks so far a quantitatively satisfactory experimental verification. Nevertheless, one has to stress that all the existing experiments [26–29] are in a qualitative agreement with the theoretical predictions.

In this paper a theory of the scaling properties of the Casimir force of a spherical model with a power-law *leading* long-ranged interaction (decreasing at long distances r as $1/r^{d+\sigma}$, with $0<\sigma\leq 2$, and $\sigma<d<2\sigma$) is presented. The results represent an extension to leading long-ranged interaction of the corresponding ones for system with short-ranged interaction [12,13]. The latter results, as we will see, can be reobtained by formally taking the limit $\sigma\to 2^-$ in the expressions pertinent to the case of long-ranged interactions.

For the kind of systems we investigate here the interaction enter the exact expressions for the free energy only through their Fourier transform which leading asymptotic behavior is $U(q)\sim a_\sigma q^{\sigma^*}$ [5,30], where $\sigma^*=\min(2,\sigma)$. As it was shown for bulk systems by renormalization group arguments $\sigma\geq 2$ corresponds to the case of subleading long-ranged interactions, i.e., the universality class then does not depend on σ [31] and coincides with that one of systems with short-ranged interactions. Values satisfying $0<\sigma<2$ correspond to leading long-ranged interactions and the critical behavior depends then on σ (see Refs. [32,33] and references therein). In the current work we will restrict ourselves to the consideration of this case only. The other case of subleading long-ranged interaction, i.e., when $\sigma>2$ is also of interest (involving, e.g., a serious modification of the standard finite-size scaling theory, see, e.g. [33–37]), but will be considered elsewhere [38]. For the current understanding of the critical behavior of finite systems with long-range interaction the interested reader is invited to consult Ref. [33].

The investigation of the Casimir effect in classical systems with long-ranged (either leading or subleading) interaction possesses some peculiarities in comparison with the short-range system. Due to the long-range character of the interaction there exists a natural attraction between the surfaces bounding the system. One easily can estimate that in systems with real boundaries (i.e., with no translational invariance) in the ordered state the L-dependent part of the excess free energy that is raised by the direct interparticle (spin) interaction is of the order of $L^{-\sigma+1}$. In the critical region one still has some effects stemming from that interaction on the background of which develops the fluctuating induced new attraction between the surfaces that is in fact the critical Casimir force. In the definition (1.1) used here, that is the common one when one considers short-range systems, these effects are superposed simultaneously. An interesting case when forces of similar origin are acting simultaneously is that one of the wetting when the wetting layer is nearly critical and intrudes between two noncritical phases if one takes into account the effect of long-range correlations and that one of the (*subleading*) long-range van der Waals forces [39–41]. In the current article we will investigate the interplay of these forces in the case of *leading* long-ranged interaction when *periodic* boundary conditions are applied, i.e., the system does not possess real boundaries. The Casimir force in systems with subleading (van der Waals type) interaction and with a broken translational invariance (i.e. possessing real physical boundaries) is a subject of investigation in a series of treatments [37,42–49]. There, in principle, one has to take into account both the long-ranged effects due to the interaction of the bounded system (say, a fluid), with the substrate of the surfaces, and, as well, the fluid-fluid long-ranged (van der Waals) interaction (which is, as a rule, treated as a short-ranged one because of the very severe technical difficulties its treatment involves). Since all these investigations were mainly concerned with the effects due to the existence of real surfaces in the system, which is not the case of a system under periodic boundary conditions, here we will not provide further details but will direct the interested reader to the references cited above and the literature cited therein.

The structure of the current article is as follows. In Sec. II we briefly describe the spherical model [which, in systems with a translational invariance, is equivalent to the $n\to\infty$ limit of the O(n) models] and give all basic expressions needed to investigate the behavior of the Casimir force. In Sec. III we derive the scaling function of the excess free energy and the Casimir force, and investigate the leading asymptotic behavior of the force both above and below the critical point. In Sec. IV we consider in some details the behavior of the force along the phase coexistence line $T<T_c, H=0$. In Sec. V we investigate the monotonicity properties of the excess free energy, and the Casimir force, and prove analytically that both the excess free energy and the force are negative for any T and H (for $\sigma>1$). The last implies that the force between the boundary surfaces of the system is always attractive. The article closes with a discussion given in Sec. VI. The technical details needed in the main text are organized in a series of Appendices.

II. THE MODEL

We consider the ferromagnetic mean spherical model with long-range interaction confined to a fully finite d-dimensional hypercubic lattice $\mathcal{L}_d$ of $N=|\mathcal{L}_d|$ sites. The model is defined by

$$\mathcal{H} = -\frac{1}{2}\sum_{ij} \mathcal{J}_{ij}\mathcal{S}_i\mathcal{S}_j - H\sum_i \mathcal{S}_i, \tag{2.1}$$

where $\mathcal{S}_i$ is the spin variable at site $i, \mathcal{J}_{ij}$ is the interaction matrix between spins at sites i and j, and H is an ordering external magnetic field. The long-wave length asymptotic form of the Fourier transform $\mathcal{J}(\boldsymbol{q})$ of the interaction potential $\mathcal{J}_{ij}$ is

$$\mathcal{J}(\boldsymbol{q}) \approx \mathcal{J}(\mathbf{0})[1-\rho_\sigma\omega_\sigma(\boldsymbol{q})], \quad |\boldsymbol{q}|\to 0, \quad \rho_\sigma>0.$$

We suppose that the interaction in the system is long-ranged with $0<\sigma<2$ implying $\omega_\sigma(\boldsymbol{q})\simeq|\boldsymbol{q}|^\sigma$. This corresponds to the inverse power-law behavior $\mathcal{J}(\boldsymbol{r})\sim r^{-d-\sigma}$, for large spin separations $r=|\boldsymbol{r}|$. The spins in the model under consideration obey the spherical constraint

$$\sum_i \langle \mathcal{S}_i^2\rangle = N, \tag{2.2}$$

where $\langle\cdots\rangle$ denotes standard thermodynamic averages taken with the Hamiltonian $\mathcal{H}$ and N is the total number of spins located at sites i of finite hypercubic lattice $\mathcal{L}_d$ of size $L_1\times L_2\times\cdots L_d=N$ (here L_i are the linear sizes of the system measured in units of the lattice constants).

Under periodic boundary conditions imposed along the finite directions of the system, the free energy density of the model is given by [5]

$$\beta\mathcal{F}_{d,\sigma}(\beta,H,\boldsymbol{L}|\boldsymbol{\Lambda}) = \frac{1}{2}\sup_{\phi>0}\left\{U_{d,\sigma}(\phi,\boldsymbol{L}|\boldsymbol{\Lambda}) + \ln\left[\frac{\beta\mathcal{J}(0)\rho_\sigma}{2\pi}\right] - \frac{\beta H^2}{\mathcal{J}(0)\rho_\sigma\phi} - \beta\mathcal{J}(0)\rho_\sigma\left(\phi+\frac{1}{\rho_\sigma}\right)\right\}, \tag{2.3a}$$

where

$$U_{d,\sigma}(\phi,\boldsymbol{L}|\boldsymbol{\Lambda}) = \frac{1}{N}\sum_{\boldsymbol{q}} \ln[\phi+\omega_\sigma(\boldsymbol{q})]. \tag{2.3b}$$

Here the vector $\boldsymbol{q}$ has the components $\{q_1,q_2,\ldots,q_d\}$ where $q_j=2\pi n_j/L_j$ and $n_j\in\{-M_j,\ldots,M_j-1\}$ with $M_j=L_j\Lambda_j/(2\pi)\gg 1$ being integer numbers, and Λ_j the cutoff at the boundaries of the first Brillouin zone along the j direction. The spherical field ϕ is introduced to ensure the fulfillment of the constraint (2.2). It is the solution of the equation

$$\beta\mathcal{J}(0)\rho_\sigma\left(1-\frac{H^2}{\phi^2\mathcal{J}^2(0)\rho_\sigma^2}\right) = \frac{1}{N}\sum_{\boldsymbol{q}} \frac{1}{\phi+\omega_\sigma(\boldsymbol{q})}. \tag{2.3c}$$

Equations (2.3a) and (2.3c) contain all the necessary information for the investigation of the critical behavior of the model under consideration.

In the bulk limit, when all the sizes of the system are infinite, the d-dimensional sums over the momentum vector $\boldsymbol{q}$ in Eqs. (2.3b) and (2.3c) transform into integrals over the first Brillouin zone. For example one has

$$U_{d,\sigma}(\phi|\Lambda) = \frac{1}{(2\pi)^d}\int_{-\Lambda}^{\Lambda} dq_1\cdots\int_{-\Lambda}^{\Lambda} dq_d \ln[\phi + \omega_\sigma(q_1,q_2,\ldots,q_d)]. \tag{2.4}$$

By analyzing the equation for the spherical field (2.3c) in the bulk limit it is easy to show that the system exhibits a phase transition for $d>\sigma$ at the critical point, β_c, given by

$$\beta_c \mathcal{J}(0)\rho_\sigma = \frac{1}{(2\pi)^d}\int_{-\Lambda}^{\Lambda} dq_1 \cdots \int_{-\Lambda}^{\Lambda} dq_d \frac{1}{\omega_\sigma(q_1,q_2\ldots,q_d)}. \tag{2.5}$$

III. SCALING FORM OF THE EXCESS FREE ENERGY AND THE CRITICAL CASIMIR FORCE

In the remainder we consider a system with a film geometry $L\times\infty^{d-1}$, which results after taking the limits $L_2\to\infty,\ldots,L_d\to\infty$ and setting $L_1=L$. For the simplicity of notations we will only consider the case when all cutoff variables are taken to be equal to each other, i.e., $\Lambda_i=\Lambda, i=1,\ldots,d$. Then $U_{d,\sigma}(\phi,\boldsymbol{L}|\boldsymbol{\Lambda})$ becomes

$$U_{d,\sigma}(\phi,L|\Lambda) = \frac{1}{L}\sum_{q_1}\frac{1}{(2\pi)^{d-1}}\int_{-\Lambda}^{\Lambda} dq_2 \cdots \int_{-\Lambda}^{\Lambda} dq_d \ln[\phi + \omega_\sigma(q_1,q_2,\ldots,q_d)]. \tag{3.1}$$

As it has been shown in [50], a sum of the above type [with $\omega_\sigma(\boldsymbol{q})\simeq|\boldsymbol{q}|^\sigma, 0<\sigma\leqslant 2$] can be evaluated using the Poisson summation formula and the identity

$$\ln(1+z^a) = a\int_0^\infty \frac{dx}{x}(1-e^{-zx})E_a(-x^a), \tag{3.2}$$

where $E_a(x)\equiv E_{a,1}(x)$, and

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)} \tag{3.3}$$

are the Mittag-Leffler functions. For a review on the properties of $E_{\alpha,\beta}(z)$ and other related to them functions, as well as for their application in statistical and continuum mechanics, see Refs. [50,51]. The properties used in the current article are summarized in Appendix A.

After some algebra for the full free energy density we get

$$\beta\mathcal{F}_{d,\sigma}(\beta,H,L) = \beta\mathcal{F}_{d,\sigma}(\beta,H) - \frac{1}{2}L^{-d}\mathcal{K}_{d,\sigma}(L^\sigma\phi), \tag{3.4}$$

where

$$\mathcal{F}_{d,\sigma}(\beta,H) \equiv \lim_{L\to\infty}\mathcal{F}_{d,\sigma}(\beta,H,L),$$

and

$$\mathcal{K}_{d,\sigma}(y) = \frac{\sigma}{(4\pi)^{d/2}}\sum_{l=1}^{\infty}\int_0^\infty dx x^{-d/2-1}\exp\left(-\frac{l^2}{4x}\right) \times E_{\sigma/2,1}(-x^{\sigma/2}y), \tag{3.5}$$

The main advantage of the above expression for the free energy, despite its complicated form in comparison to Eq. (2.3a), is the simplified dependence on the size L which now enters only *via* the arguments of some functions. This gives the possibility, as it is explained below, to obtain the scaling functions of the excess free energy and the Casimir force. It is worthwhile noting that under a sharp cutoff Λ a special care has to be taken when performing finite-size scaling calculations in order to avoid receiving artificial, i.e., not existing in real systems, finite-size Λ-dependent contributions. This question is considered in detail in [37]. In obtaining Eq. (3.4) the suggested there recipe has been applied [see Eq. (27) in [37] and the discussion devoted to it]. According to these findings, for the finite-size contributions in the following we are going to send the cutoff to infinity.

In Eq. (3.4), ϕ is the solution of the corresponding spherical field equation that follows by requiring the partial derivative of the right-hand side of Eq. (3.4) with respect to ϕ to be zero. Let us denote the solution of the corresponding bulk spherical equation by ϕ_∞. Then, for the excess free energy (per unit area) it is possible to obtain from Eqs. (1.2) and (3.4) the finite size scaling form, valid for $\sigma<d<2\sigma$,

$$f^{\mathrm{ex}}(\beta,H,L|d) = \beta L^{-(d-1)}X_f(x_1,x_2), \tag{3.6}$$

with scaling variables

$$x_1 = (\beta-\beta_c)\mathcal{J}(0)\rho_\sigma L^{1/\nu} \tag{3.7a}$$

and

$$x_2 = HL^{\Delta/\nu}\sqrt{\beta/\mathcal{J}(0)\rho_\sigma}. \tag{3.7b}$$

Here $\nu=1/(d-\sigma)$ and $\Delta=(d+\sigma)/[2(d-\sigma)]$ are the critical exponents of the spherical model (for $\sigma<d<2\sigma$, and $0<\sigma\leqslant 2$). Notice that, according to the definitions (3.7a) and (3.7b), the subcritical region $T<T_c$ corresponds to positive values of x_1. In Eq. (3.6) the universal scaling function $x^{\mathrm{ex}}(x_1,x_2)$ of the excess free energy has the form

$$X_f(x_1,x_2) = -\frac{1}{2}x_2^2\left(\frac{1}{y_L}-\frac{1}{y_\infty}\right) - \frac{1}{2}x_1(y_L-y_\infty) - \frac{\sigma}{2d}|D_{d,\sigma}|(y_L^{d/\sigma}-y_\infty^{d/\sigma}) - \frac{1}{2}\mathcal{K}_{d,\sigma}(y_L), \tag{3.8}$$

where the $y_L=\phi_L L^\sigma, y_\infty=\phi_\infty L^\sigma$, and

$$D_{d,\sigma} = 2\pi\left[(4\pi)^{d/2}\Gamma\left(\frac{d}{2}\right)\sigma\sin\left(\frac{\pi d}{\sigma}\right)\right]^{-1}. \tag{3.9}$$

In Eq. (3.8) y_L is the solution of the spherical field equation for the finite system obtained by minimizing the free energy with respect to y_L

$$x_1 = \frac{x_2^2}{y_L^2} - |D_{d,\sigma}|y_L^{d/\sigma-1} - \frac{\partial}{\partial y_L}\mathcal{K}_{d,\sigma}(y_L). \tag{3.10}$$

For the infinite system the corresponding equation is

$$x_1 = \frac{x_2^2}{y_\infty^2} - |D_{d,\sigma}|y_\infty^{d/\sigma-1}. \tag{3.11}$$

According to Eq. (1.1), the finite-size scaling function of the Casimir force for the system under consideration is

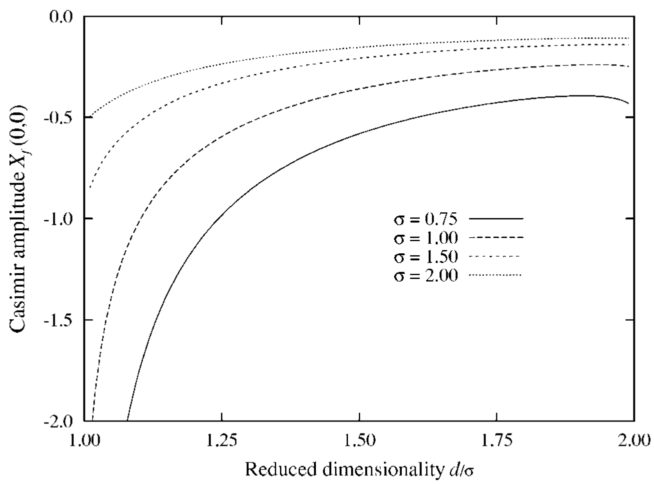

FIG. 1. Behavior of the Casimir amplitude as a function of d.

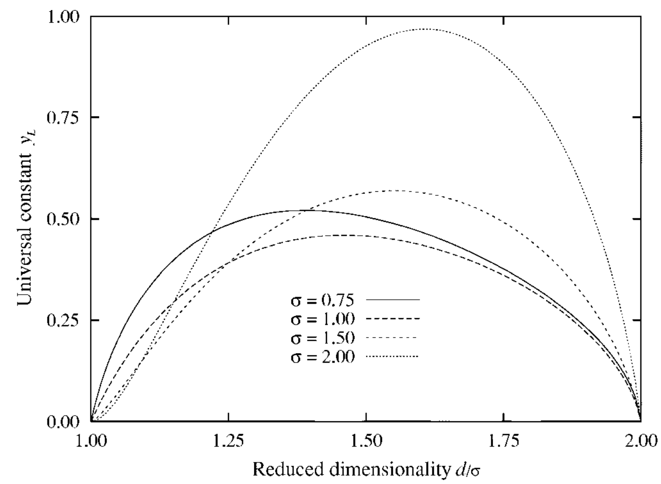

FIG. 2. Behavior of the scaling variable y_L as a function of d at the critical point $T=T_c$. We recall that the finite-size correlation length ξ_L is related to y_L via $\xi_L = L y_L^{-1/\sigma}$ [5].

$$X_{\mathrm{Casimir}}(x_1,x_2) = \frac{\sigma+1}{2}x_2^2\left(\frac{1}{y_L}-\frac{1}{y_\infty}\right) - \frac{\sigma-1}{2}x_1(y_L-y_\infty) - \frac{\sigma(d-1)}{2d}|D_{d,\sigma}|\left(y_L^{d/\sigma}-y_\infty^{d/\sigma}\right) - \frac{1}{2}(d-1)\mathcal{K}_{d,\sigma}(y_L). \tag{3.12}$$

Note that in the limit $\sigma \to 2^-$ Eqs. (3.6)–(3.12) reproduce exactly the corresponding ones for the case of the short-range interaction [12,13,15]. In such a case the above equations simplify greatly since then $E_{1,1}(z)=\exp(z)$, and the function $\mathcal{K}_{d,\sigma}(y)$ defined in Eq. (3.5) becomes

$$\mathcal{K}_{d,2}(y) = \frac{4}{(2\pi)^{d/2}}y^{d/4}\sum_{l=1}^{\infty} l^{-d/2}K_{d/2}(l\sqrt{y}), \tag{3.13}$$

where K_ν is the modified Bessel function.

In the present article we will concentrate on the investigation of the behavior of the Casimir force and the excess free energy in different regions of the phase diagram. We will also evaluate some critical amplitudes for selected values of the parameters d and σ. The analysis will be done analytically for the cases where one can obtain simple expressions and is then extended numerically to cases which are not accessible by analytical means.

First, let us note that when the interaction becomes more long-ranged, i.e., σ decreases, the finite-size corrections due to the direct interaction between the surfaces delimiting the system becomes stronger implying an increase of the modulus of the Casimir amplitude $X_f(0,0)$. In Fig. 1 we present the numerical evaluation of the Casimir amplitudes as a function of d for some selected values of σ. The results show that the amplitude is indeed an increasing function of d at fixed σ, and an increasing function of σ at a fixed d. Note also that in accordance with the general expectations, the amplitudes are negative.

In order to obtain the amplitudes, one needs to know the value of $y_L(T)$ at the critical point $T=T_c$ that is the solution of the equation for the spherical field (3.10). These results have their own important physical meaning. We recall that y_L is directly connected to the finite-size correlation length $\xi_L = L y_L^{-1/\sigma}$ of the system [5]. The results for $y_L(T_c)$ are shown in Fig. 2.

In Fig. 3 we present our results for the Casimir force evaluated at the bulk critical point of the model as a function of d for some selected values of σ. We observe that the Casimir force behaves in a different way depending on whether σ is smaller or larger than $\sigma=1$. For $\sigma \leqslant 1$ it is decreasing monotonically as a function of d, while for $\sigma > 1$ it is not.

In the following we turn our attention to the investigation of the thermodynamic functions of interest as a function of the scaling variable x_1 for fixed d and σ. Let us first consider the situations where it is possible to obtain some results analytically.

Let us first consider the asymptotic forms of the excess free energy and the Casimir force in the subcritical region (i.e., $T \lesssim T_c$). Taking into account that then (i.e., when $x_1 \gg 1, x_2=0$), according to Eqs. (3.10) and (3.11) $y_L \to 0^+, y_\infty$

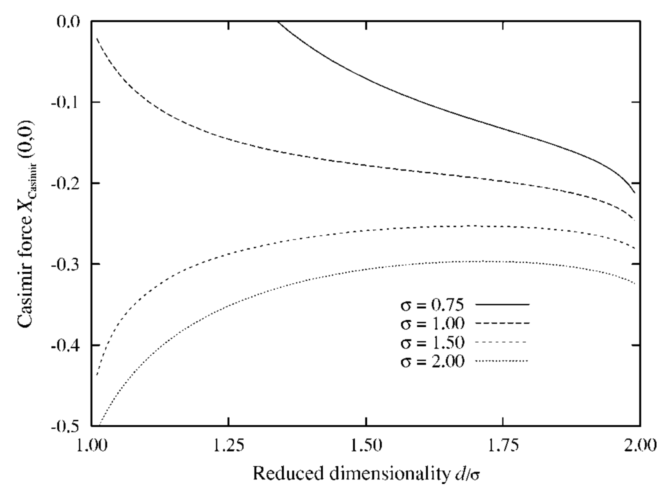

FIG. 3. The behavior of the Casimir force at $T=T_c$ as a function of d.

=0, as well as the asymptotic (B13) of the function $\mathcal{K}_{d,\sigma}(y_L)$ for small values of the argument (derived in Appendix B) it is easy to see that below the critical temperature

$$X_f(x_1 \to \infty, 0) \simeq -\frac{\sigma}{2\pi^{d/2}}\Gamma\left(\frac{d}{2}\right)\zeta(d), \tag{3.14}$$

and

$$X_{\text{Casimir}}(x_1 \to \infty, 0) \simeq -\frac{\sigma(d-1)}{2\pi^{d/2}}\Gamma\left(\frac{d}{2}\right)\zeta(d). \tag{3.15}$$

The above results reflect the dominating contribution of the Goldstone modes in the subcritical-regime of an O(n) model—both the excess free energy and the Casimir force scaling functions do not tend exponentially-in-L to zero, but to *finite* constants. For $\sigma=2$ these constants coincide with those known from short-range systems (see, e.g. [15] and the references cited therein). Note also that, in contrast with systems with real boundaries, the direct interspin long ranged interaction below T_c does not lead to a $L^{-(\sigma-1)}$ contribution, which is well known from studies of van der Waals systems exhibiting wetting phase transitions [39,52]. This is due to the application of periodic boundary conditions, i.e., the system under consideration lacks real physical boundaries.

Let us consider the critical behavior of the force for $T > T_c$ in a bit more details. Then, when $x_2=0$ and $x_1 \to -\infty$ from Eqs. (3.10) and (3.11) one obtains $y_L \simeq y_\infty(1+\varepsilon_{d,\sigma})$, where

$$\varepsilon_{d,\sigma} = \frac{a_{d,\sigma}}{\left(\frac{d}{\sigma}-1\right)|D_{d,\sigma}|y_\infty^{d/\sigma+1}}, \tag{3.16}$$

and

$$y_\infty = \left(\frac{|x_1|}{|D_{d,\sigma}|}\right)^{a/(d-\sigma)}. \tag{3.17}$$

Therefore, the leading behavior of the scaling function of the force in that region is

$$X_{\text{Casimir}} \simeq -A_{d,\sigma}y_\infty^{-1} \simeq -A_{d,\sigma}[(\beta_c - \beta)\mathcal{J}(0)\rho_\sigma/|D_{d,\sigma}|]^{-\sigma/(d-\sigma)}L^{-\sigma}, \tag{3.18}$$

where

$$A_{d,\sigma} = \frac{a_{d,\sigma}}{2}(\sigma+d-1). \tag{3.19}$$

Equation (3.18), valid for $0<\sigma<2$, implies that above $T_c, F_{\text{Casimir}} \simeq -X_+|t|^{-\gamma}L^{-(d+\sigma)}$, with $\gamma=\sigma/(d-\sigma)$, and $X_+>0$, i.e., the force remains attractive and decays in a power-in-L and not in an exponentially-in-L way, as it is in systems with short ranged interactions. This behavior is in full correspondence with the long-ranged character of the interaction. Similar also, as it has been recently established, is the behavior of the Casimir force and the excess free energy in systems with van der Waals type interaction [37] (see also [38]), despite that their critical exponents are those of the short-ranged systems.

The obtained analytical results are supported by numerical analysis of the expressions for the scaling functions of the excess free energy and the Casimir force at zero external field. The corresponding data is presented in Fig. 4 (for the excess free energy) and in Fig. 5 (for the Casimir force). While the scaling function of the excess free energy is monotonic regardless of the used values of d and σ, the behavior of Casimir force depends strongly on the range of the interaction σ. For $\sigma>1$ it is monotonically increasing as it can be seen from the case $\sigma=2$, corresponding to short range interaction, and the long-range case with $\sigma=1.5$. For $\sigma=1$ the monotonicity changes and $X_{\text{Casimir}}(x_1,0)$ becomes decreasing for values of $\sigma<1$. As example we show its behavior for $\sigma=0.75$.

We close this section by presenting the outcome of the numerical analysis of the behavior of the scaling functions of the excess free energy, shown in Fig. 6, and that of the Ca-

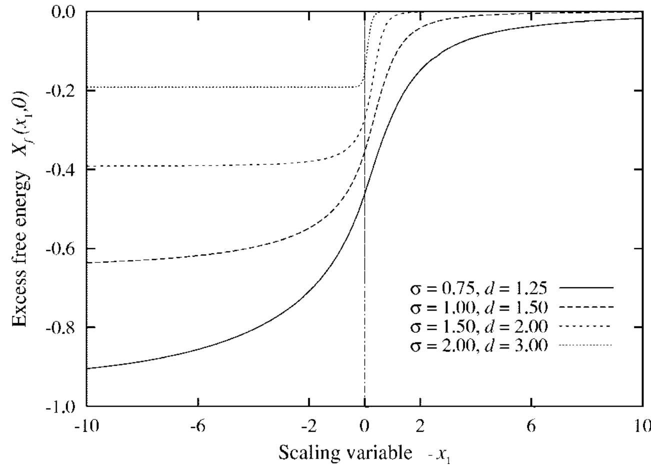

FIG. 4. The universal finite-size scaling function of the excess free energy $X_f(x_1,0)$ from Eq. (3.8) as a function of $-x_1 \sim (T-T_c)L^{1/\nu}$, for some selected values of σ at zero external magnetic field. One observes that, in full accordance with the corresponding statement from Sec. V, $X_{\text{Casimir}}(x_1,0)$ is a monotonically increasing function of the temperature T regardless of value of σ.

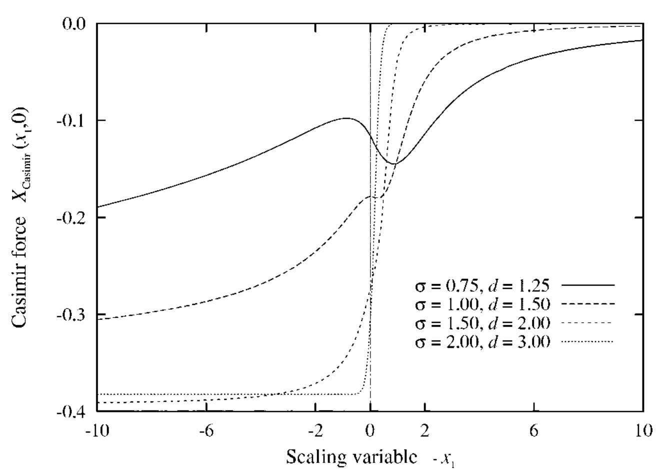

FIG. 5. The universal finite-size scaling function of the Casimir force $X_{\text{Casimir}}(x_1,0)$ as a function of the scaling variable $-x_1 \sim (T-T_c)L^{1/\nu}$, at zero external magnetic field $H=0$. One observes that, in full accordance with the corresponding statement from Sec. V, $X_{\text{Casimir}}(x_1,0)$ is a monotonically increasing function of the temperature T (for $\sigma>1$) and possesses a complex behavior for $\sigma \leqslant 1$.

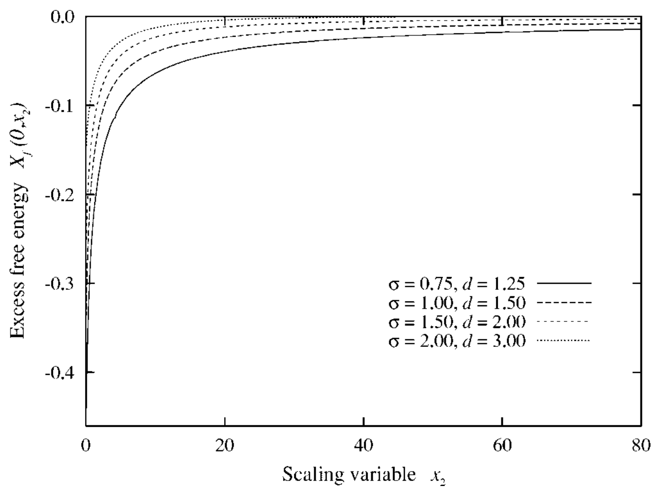

FIG. 6. The universal finite-size scaling function of the excess free energy $X_f(0,x_2)$, for some values of σ, as a function of the scaling variable $x_2 \sim HL^{\Delta/\nu}$ at the bulk critical point $T=T_c$. One observes $X_f(0,x_2)$ is a monotonically increasing function of the field H for arbitrary σ.

simir force, shown in Fig. 7, as a function of the scaling variable x_2 at the bulk critical temperature. One observes that the excess free energy is a monotonically increasing function of the external magnetic field H independently of the range of the interaction. However the Casimir force is a *nonmonotonic* function of H and *has a minimum at* $x_2 \neq 0$ which depth depends of the parameter σ. The minimum is found to be at $x_2=0.084$, 0.145, 0.263, and 0.416 for $\sigma=2$, 1.5, 1, and 0.75, respectively. So, as long as σ goes smaller the minimum becomes deeper. Indeed the ratio of the Casimir force evaluated at the minimum to its value at $H=0$ is a decreasing function of σ. It is given by 1.017, 1.073, 1.215, and 1.513 for $\sigma=2$, 1.5, 1, and 0.75, respectively.

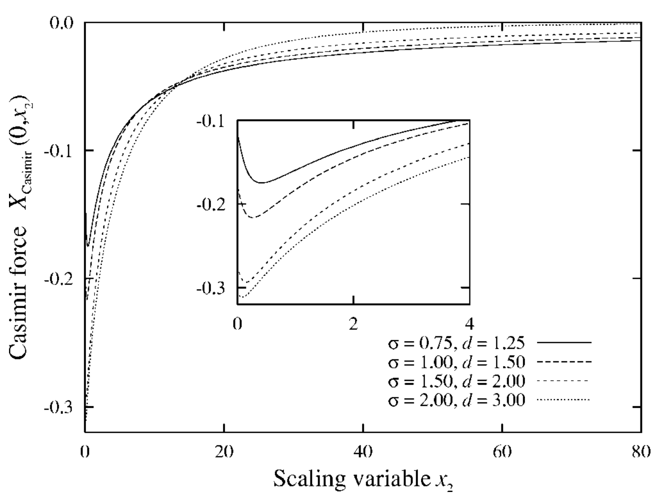

FIG. 7. The universal finite-size scaling function of the Casimir force X_{Casimir}, for some values of σ, as a function of the scaling variable $x_2 \sim HL^{\Delta/\nu}$ at the bulk critical temperature $T=T_c$. One observes that $X_{\text{Casimir}}(0,x_2)$ is not a monotonically increasing function of the field H for all values of $\sigma \leqslant 2$ including the short-range case.

IV. CASIMIR (SOLVATION) FORCE ALONG THE PHASE COEXISTENCE LINE

Here we investigate the behavior of the Casimir force along the line $H=0$ when $T<T_c$. This is a line of a first order phase transition with respect to the magnetic field H. The finite-size rounding of the first-order transitions in O(n) models has been already studied by Fisher and Privman in [53] for a fully finite and cylinder geometries. Later their predictions have been verified in details for the spherical model system with such a geometry in [54], while in [55,56] their arguments have been extended to a geometry of the type $L^{d-d'} \times \infty^{d'}$, where d' has been chosen so that no phase transition of its own exists in the finite system, i.e., $d'<\sigma$ has been supposed. Here we extend these investigations to cover also the cases $d'=\sigma$ and $d'>\sigma$ in systems with a film geometry, i.e., when $d'=d-1$. We will be only interested in the behavior of the Casimir force.

For $T<T_c$ and small H Eqs. (3.6)–(3.12) are still valid, but there the limit $y_L \ll 1$ has to be taken (i.e., we suppose that $x_1 \gg x_2^2$). As it is clear from Eq. (B13), then there are three subcases to be considered.

(i) *The case* $d-1<\sigma$. Then in the finite system there is no phase transition on its own. For the excess free energy one obtains

$$f^{\text{ex}}(\beta,H) = -\frac{\sigma}{2\pi^{d/2}}\Gamma(d/2)\zeta(d)L^{-(d-1)} + \beta m_0 HL \times\left\{1-\frac{1}{2}\left(\frac{m_0}{m_L}+\frac{m_L}{m_0}\right)+\frac{\sigma}{2(d-1)}\left(\frac{m_0}{m_L}-\frac{m_L}{m_0}\right)\right\}, \tag{4.1}$$

where

$$\frac{m_L}{m_0} = \sqrt{\left[\frac{|D_{d-1,\sigma}|}{2x_m}\right]^2+1} - \frac{|D_{d-1,\sigma}|}{2x_m}. \tag{4.2}$$

Here $m_L=H/[\rho_\sigma \mathcal{J}(0)\phi]$ is the magnetization of the finite system, $m_0=\sqrt{1-T/T_c}$ is the spontaneous magnetization, and $x_m=\beta m_0(T)L\xi_L^{d-1}H$, which has the meaning of the ratio of the total magnetic energy in the correlated volume $V_{\text{cor}}=L\xi_L^{d-1}$ to the thermal energy k_BT per degree of freedom, is the scaling variable. (We recall that in the spherical model the true finite-size correlation length ξ_L is equal to $\phi^{-1/\sigma}$ [5,56].) Next, it is easy to see from Eq. (4.2) that $x_m=O(1)$ involves $H=O(L^{-\sigma/(1+\sigma-d)})$, that is the scale on which the jump in the bulk magnetization is rounded off. From this observation and from Eq. (4.1) one obtains that the H dependent correction to the Casimir force is then of the order of $L^{-\sigma/(1+\sigma-d)}$. [Note that $\sigma/(1+\sigma-d)>d$ for $d>\sigma$, and, so, the term proportional to H in Eq. (4.1) will indeed contribute as a correction towards the Casimir force.]

(ii) *The case* $d-1=\sigma$. This is the borderline case between the one when in the finite system there is no phase transition of its own (for $d-1<\sigma$) and the one in which in the finite system there is such a phase transition (for $d-1>\sigma$). In this case an essential singular point exists in the finite-size system at $T=H=0$. For the excess free energy one now obtains

$$f^{\mathrm{ex}}(\beta,H) = -\frac{\sigma}{2\pi^{d/2}}\Gamma(d/2)\zeta(d)L^{-(d-1)} + \beta m_0 HL\left\{1-\frac{m_0}{m_L}\right\}, \tag{4.3}$$

where

$$\frac{m_L}{m_0} = \sqrt{\left[\frac{1}{(4\pi)^{\sigma/2}\Gamma(\sigma/2)}\frac{1}{\overline{x_m}}\right]^2+1} + \frac{1}{(4\pi)^{\sigma/2}\Gamma(\sigma/2)}\frac{1}{\overline{x_m}} \tag{4.4}$$

and $\overline{x_m}=\beta m_0(T)HL\xi_L^{d-1}/\ln(L/\xi_L)$. The above equations are to be compared with the previous case. One observes, that the main difference is the existence of logarithmic-in-L dependence that is introduced via the scaling field variable $\overline{x_m}$. As a result the rounding of the jump in the magnetization takes place on a scale given by $H=L^{-\sigma}\exp(-\text{const } L)$, i.e., the scale in this case is exponentially small in L.

(iii) *The case* $d-1>\sigma$. In this case there is a true phase transition of its own in the finite system at some $T_{c,L}=T_c-\varepsilon L^{-1/\nu}$, i.e., no rounding of the jump of the magnetization is possible. One only observes L-dependent corrections of the finite-size magnetization m_L with respect to the spontaneous magnetization m_0. One finds that the crossover from d to $d-1$ critical behavior happens at $T_{c,L}$ with

$$\varepsilon = \frac{\pi^{(d-1)/2}}{(2\pi)^{\sigma}}\frac{C_{d,\sigma}}{\Gamma(d/2)}\frac{1}{\beta_c\mathcal{J}(0)\rho_\sigma}, \tag{4.5}$$

and, when $|H|L^{\sigma}\ll 1$,

$$f^{\mathrm{ex}}(\beta,H) = -\frac{\sigma}{2\pi^{d/2}}\Gamma(d/2)\zeta(d)L^{-(d-1)} + \beta m_0 HL\frac{a}{L^{d-\sigma}}, \tag{4.6}$$

with

$$a = \frac{\pi^{(d-1)/2}}{2(2\pi)^{\sigma}}\frac{C_{d,\sigma}}{\Gamma(d/2)}\frac{1}{\beta m_0^2(T)\mathcal{J}(0)\rho_\sigma}, \tag{4.7}$$

and $m_L\simeq m_0(1-a/2)$.

Finally, we would like to note that in O(n) systems one observes for $T<T_c$ in addition to the rounding of the jump of the order parameter also rounding of the spin wave singularities. According to the general theory [53,54], their rounding occurs on the scale for which $x_s=|H|L^{\sigma}=O(1)$. As it is clear from Eq. (3.7) [and taking into account that if $T<T_c$ one can rewrite x_1 as $x_1=\beta m_0(T)^2\rho_\sigma\mathcal{J}(0)$, with $x_1\gg 1$] the scale on which the rounding of the spin wave singularities sets in involves that $x_1\sim x_2^2$ there. Then, in this regime, the solution of the spherical field equations for the finite and the infinite system (3.10) and (3.11) will be again $y_L=O(1)$ and $y_\infty=O(1)$. Since x_1 and x_2 can be expressed from Eqs. (3.10) and (3.11) in terms of y_L and y_∞, we conclude that, according to Eq. (3.12), in the regime in which the spin waves are of importance, the Casimir force will be $F_{\mathrm{Casimir}}=O(L^{-d})$, possessing a nontrivial H dependence. If one would like to reveal more on this dependence the numerical treatment is unavoidable. Note that when the field is strong enough to suppress the spin-wave excitations, i.e., when $x_s\gg 1$ and $T<T_c$, one will have an Ising-like system. In this regime $y_L\gg 1, y_\infty\gg 1$, and the Casimir force will be of the order of $L^{-(d+\sigma)}$ [see Eq. (3.18)] under periodic boundary conditions. (If the system was possessing real bounding surfaces like, say, under Dirichlet-Drichlet boundary conditions, one would expect that the corresponding contribution in the force is of the order of $L^{-\sigma}$.)

V. MONOTONICITY PROPERTIES OF THE EXCESS FREE ENERGY AND THE CASIMIR FORCE

Let us denote by $g_L(x_2,y)$ and $g_\infty(x_2,y)$ the right-hand side of Eqs. (3.10) and (3.11), respectively. Now we prove that (i) $g_L(x_2,y)>g_\infty(x_2,y)$ and (ii) that $g_L(x_2,y)$ and $g_\infty(x_2,y)$ are monotonically decreasing functions of y.

(i) First, let us note that $E_{\alpha,\beta}(-x)$ is a completely monotonic function of $x\geq 0$ [57–60] for $0<\alpha\leq 1$ and $\beta\geq\alpha$. (In [57] this property was shown to hold for $E_{\alpha,1}(-x)\equiv E_\alpha(-x)$ and was later extended to $E_{\alpha,\beta}(-x)$ in [58] and [59]; see also [60].) This means that for all $n=0, 1, 2, 3, \ldots$ one has

$$(-1)^n\frac{d^nE_{\alpha,\beta}(-x)}{dx^n}\geq 0, \quad x\geq 0, \quad 0<\alpha\leq 1, \quad \beta\geq\alpha. \tag{5.1}$$

Then, from $n=0$ it immediately follows that $E_{\alpha,\alpha}(-x)>0$ when $x\geq 0$. Now, from Eqs. (3.10) and (3.11), it immediately follows that $g_L(x_2,y)>g_\infty(x_2,y)$.

(ii) The required property follows from the monotonicity of the function $E_{\alpha,\alpha}(-x)$ for $x\geq 0$ and the explicit form of the right-hand sides of Eqs. (3.10) and (3.11).

Having proved (i) and (ii), it is easy to understand now that for any given values x_1 and x_2 of the scaling variables the solution of the spherical field equation for the finite system will be larger than that for the infinite system, i.e., $y_L(x_1,x_2)>y_\infty(x_1,x_2)$. (Since the correlation lengths in the finite and the infinite system are $\xi_L=y_L^{-1/\sigma}$ and $\xi_\infty=y_\infty^{-1/\sigma}$ [5], correspondingly, the physical meaning of the above result is that the correlation length of the finite system is always smaller than that of the infinite one.) We are then ready to prove the following.

(A) *For* $x_1\geq 0$ *and* $x_2=0$ *the excess free energy scaling function is negative, i.e.,* $X_f(x_1\geq 0,x_2=0)<0$.

(B) *The excess free energy scaling function* $X_f(x_1,x_2)$ *is a monotonically increasing function of the temperature T and the magnetic field* $|H|$.

Let us start with statement (A).

(A) From the explicit form of the Eq. (3.8) it is clear that the statement (A) will be true if $E_{\alpha,1}(-x)\geq 0$ when $x\geq 0$. The last follows from Eq. (5.1) for $n=0$, and, thus, $X_f(x_1,x_2)<0$.

Let us now prove the statement (B).

(B) From Eq. (3.8) one obtains

$$\frac{\partial X_f}{\partial x_1} = \frac{1}{2}(y_\infty - y_L) < 0, \tag{5.2}$$

and

$$\frac{\partial X_f}{\partial x_2} = x_2\left(\frac{1}{y_\infty}-\frac{1}{y_L}\right) > 0. \tag{5.3}$$

Equation (5.2) implies that $X_f(x_1,x_2)$ *is a monotonically increasing function of* T, whereas Eq. (5.3) states that *it is a monotonically increasing function of* $|H|$ too.

Using (B) one can now prove the following.

(C) *The excess free energy scaling function is negative for any* T *and* H, *i.e.*, $X_f(x_1,x_2)<0$ *for any* x_1 *and* x_2.

Indeed, from the monotonicity property (B) and from (A) it is clear that in order to prove (C) it is enough to show that it holds for values of T *above* T_c, i.e., when $y_L \gg 1$ and $y_\infty \gg 1$. Then, from Eqs. (3.10) and (3.11) and the asymptotic (B14) one obtains $y_L = y_\infty(1+\varepsilon), 0<\varepsilon \ll 1$, where

$$\varepsilon = \frac{a_{d,\sigma}}{y_\infty^2\left[2\frac{x_2^2}{y_\infty^2} + |D_{d,\sigma}|y_\infty^{d/\sigma-1}\left(\frac{d}{\sigma}-1\right) + 2\frac{a_{d,\sigma}}{y_\infty^2}\right]}. \tag{5.4}$$

Next, from Eq. (3.8) it follows that

$$X_f(x_1,x_2) \simeq -\frac{1}{2}\frac{a_{d,\sigma}}{y_\infty}(1-\varepsilon) < 0. \tag{5.5}$$

Thus the excess free energy is indeed always negative.

Finally, we prove that the following.

(D) *For* $\sigma \geq 1$ *the Casimir force is always negative, i.e., it is a force of attraction between the surfaces bounding the system*.

We start by multiplying Eq. (3.11) with y_∞ and Eq. (3.10) with y_L, and then adding the results together. One obtains

$$x_1(y_L - y_\infty) = x_2^2\left(\frac{1}{y_L} - \frac{1}{y_\infty}\right) - |D_{d,\sigma}|(y_L^{d/\sigma} - y_\infty^{d/\sigma}) - y_L\frac{d}{dy_L}\mathcal{K}_{d,\sigma}(y_L). \tag{5.6}$$

Inserting the above expression in Eq. (3.12), one obtains

$$X_{\mathrm{Casimir}}(x_1,x_2) = x_2^2\left(\frac{1}{y_L} - \frac{1}{y_\infty}\right) - \frac{1}{2}\left(1-\frac{\sigma}{d}\right)|D_{d,\sigma}|(y_L^{d/\sigma} - y_\infty^{d/\sigma}) - \frac{1}{2}(d-1)\mathcal{K}_{d,\sigma}(y_L) + \frac{\sigma-1}{2}y_L\frac{d}{dy_L}\mathcal{K}_{d,\sigma}(y_L). \tag{5.7}$$

Since, according to what already has been proven, $y_L > y_\infty$, and $\mathcal{K}_{d,\sigma}(y_L)$ is a positive and monotonically decreasing function of y_L [the last follows from the explicit form of $\mathcal{K}_{d,\sigma}(y_L)$ given in Eq. (3.5) and the property (5.1) of $E_{\alpha,1}(x)$ for $n=0$ and $n=1$], from the above expression one immediately confirms the validity of statement. *(D)*. In addition, from Eq. (3.12) it is easy to see that $X_{\mathrm{Casimir}}(x_1,x_2)<0$ also for $\sigma<1$ if $x_1 \leq 0$, i.e., for $T \geq T_c$. Furthermore, from Eqs. (1.5) and (3.12) it follows that

$$\frac{\partial}{\partial x_1}X_{\mathrm{Casimir}}(x_1=0,x_2=0) = -\frac{\sigma-1}{2}y_{L,c}, \tag{5.8}$$

where from we conclude, that at $T=T_c$ the Casimir force is an increasing function of T for $\sigma>1$ (see Fig. 5), and a decreasing function of T when $\sigma<1$ (see Fig. 5). Therefore, at the critical point the monotonicity of the force changes as a function of σ at $\sigma=1$ where we have an inflexion point.

VI. DISCUSSION

In the current article we consider the behavior of the excess finite-size free energy and the Casimir (solvation) force in a classical system with leading long range interactions in the limit $n\to\infty$ of the O(n) models (i.e., within the spherical model). In this limit, the model has the peculiarity of being exactly solvable and in the same time the ability to describe in a convincing way the basic features of the physical behavior of O(n) systems with finite number of component spins n. This is very useful if one later tries to investigate more realistic models using either numerical (say Monte Carlo) or more elaborate analytical techniques. Furthermore, as it has been already pointed out in the Introduction the scaling functions $X_{\mathrm{Casimir}}^{(p)}(x)/n$ of the Casimir force for the 3D Ising, XY, Heisenberg and Spherical models with short-ranged interactions practically coincide [15] if $x=L/\xi \gtrsim 2$, where ξ is the true bulk correlation length. One might expect the same to be true also for the case of leading long-ranged interactions in such systems.

In the current treatment the dimensionality d of the models and the parameter controlling the range of the interaction σ are chosen so, that the hyperscaling is kept valid, i.e., $\sigma<d<2\sigma$ is supposed. In this regime the critical exponents depend on σ. We demonstrate that, despite of the choice of σ, the excess free energy scaling function X_f (see Fig. 4 and Fig. 6) is a monotonic function of the temperature T and the magnetic field H, with X_f being always a negative function. Surprisingly, to a given extend, the above properties do not hold in such a general fashion for the Casimir (solvation) force (see Fig. 5 and Fig. 7). This is in line with the results of Sec. V where we show analytically that the force is attractive for any T and $\sigma \geq 1$, as well as for any $T \geq T_c$ if $\sigma<1$. The monotonicity of the force turns out to depend on σ. For example, if $\sigma>1$ at $T=T_c$ and $H=0$ the force is an increasing function of T and L^{-1}, while for $\sigma<1$ it is a decreasing function of both T and L^{-1} at this point [see Eq. (5.8) and Fig. 5]. In addition, one derives that for $T=T_c$ the minimum of the force is *not* at $H=0$ (see Fig. 7). Indeed, at $T=T_c$ the minimum has been found to be at some *finite* value of the scaling field variable $x_2 \sim HL^{\Delta/\nu}$. For $\sigma=2$, 1.5, 1, and 0.75 the minimum at $T=T_c$ is found to be at $x_2 \simeq 0.084$, 0.145, 0.263, and 0.416, respectively. Such an occurrence of a force minimum for a nonzero bulk field has also been reported for the case of $(+,+)$ boundary conditions [46,47]. Here, in this Section, we provide more details for the universal finite-size scaling function of the Casimir force $X_{\mathrm{Casimir}}(x_1,x_2)$ presenting the numerical results for it as a function of both x_1 and $x_2 \geq 0$ in Fig. 8. There the effects due to both the temperature and the magnetic field are demonstrated [we recall that $x_1 \sim (T-T_c)L^{1/\nu}, x_2 \sim HL^{\Delta/\nu}$]. We observe that for $T<T_c$ and $H\neq 0$ a valley shows up in the vicinity of the critical temperature that disappears for temperatures far away from the critical point. More precisely, one observes that there exists a finite value x_1^* of x_1, such that for any $x_1^* > x_1 \geq 0$ there is a local minimum of the force at some finite $x_{2,\min}$, i.e., at $H \neq 0$. For $x_1 > x_1^*$ there is no such minimum at nonzero H. In Fig. 8 the last is shown for the cases $\sigma=0.75$, 1, 1.5 and $\sigma=2$ (the short-range case). Note, that for $\sigma=0.75$ one needs

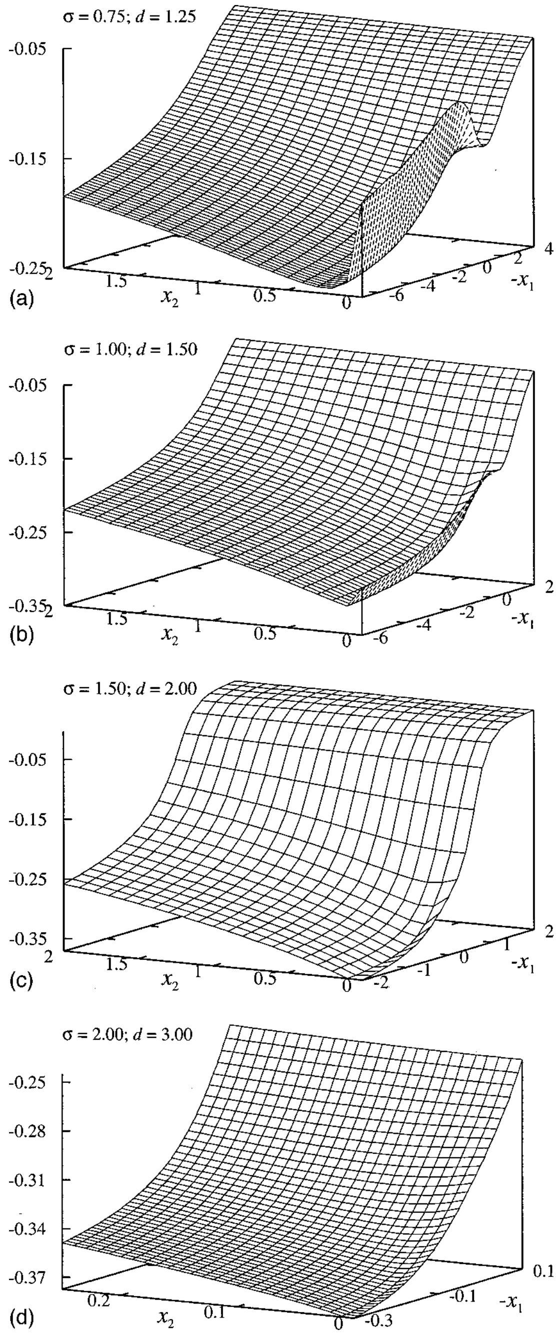

FIG. 8. The universal finite-size scaling function of the Casimir force as a function of scaling variables x_1 and x_2 for some values of the parameter σ and the corresponding values of d. The visualization is limited to positive values of x_2 since the function is even in H.

to go deeply in the subcritical region to find out where exactly the valley vanishes. In the short-range case $\sigma=2$ we established that $x_1^* \simeq 0.28$.

ACKNOWLEDGMENTS

H.C. acknowledges financial support from the Associateship Scheme of the Abdus Salam International Centre of Theoretical Physics (ICTP), Trieste, Italy. D.D. acknowledges the hospitality of Max-Planck-Institute for Metals Research in Stuttgart and the partial support via Project F-1402 of the Bulgarian Fund for Scientific Research.

APPENDIX A: SOME PROPERTIES OF THE MITTAG-LEFFLER TYPE FUNCTIONS

The Mittag-Leffler type functions are defined by the power series [51]:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha,\beta > 0. \tag{A1}$$

They are entire functions of finite order of growth. The functions are named after Mittag-Leffler who first considered the particular case $\beta=1$. These function are very popular in the field of fractional calculus (for a recent review see Ref. [51]).

One of the most useful property of these functions is the identity [51]

$$\frac{1}{1+z} = \int_0^{\infty} dx\, e^{-x} x^{\beta-1} E_{\alpha,\beta}(-x^{\alpha} z), \tag{A2}$$

which is obtained by means of term-by-term integration of the series (A1). The integral in Eq. (A2) converges in the complex plane to the left of the line $\mathrm{Re}\, z^{1/\alpha}=1, |\arg z| \leqslant \frac{1}{2}\alpha\pi$. The identity (A2) lies in the basis of the mathematical investigation of finite-size scaling in the spherical model with algebraically decaying long-range interaction (see Ref. [5] and references therein).

In some particular cases the functions $E_{\alpha,\beta}(z)$ reduce to known functions. For example, in the case corresponding to the short range case we have

$$E_{1,1}(z) = \exp(z). \tag{A3}$$

Setting $z=y^{-\alpha}, y>0$, and $x=ty$, we obtain the Laplace transform

$$\frac{y^{\alpha-\beta}}{1+y^{\alpha}} = \int_0^{\infty} dt\, e^{-yt} t^{\beta-1} E_{\alpha,\beta}(-t^{\alpha}) \tag{A4}$$

from which we derive the useful identity

$$\frac{1}{1+z^{\alpha}} = \int_0^{\infty} dx \exp(-xz) x^{\alpha-1} E_{\alpha,\alpha}(-x^{\alpha}), \tag{A5}$$

by setting $\beta=\alpha$.

The asymptotic behavior for large arguments of the Mittag-Leffler functions is given by the Lemma [61].

Let $0<\alpha<2, \beta$ be an arbitrary complex number, and γ be a real number obeying the condition

$$\frac{1}{2}\alpha\pi < \gamma < \min\{\pi, \alpha\pi\}.$$

Then for any integer $p \geqslant 1$ the following asymptotic expressions hold when $|z| \to \infty$.

At $|\arg z| \leqslant \gamma$,

$$E_{\alpha,\beta}(z) = \frac{1}{\alpha} z^{(1-\beta)/\alpha} e^{z^{1/\alpha}} - \sum_{k=1}^{\infty} \frac{z^{-k}}{\Gamma(\beta - \alpha k)} + \mathrm{O}(|z|^{-p-1}). \tag{A6}$$

At $\gamma \leqslant |\arg z| \leqslant \pi$,

$$E_{\alpha,\beta}(z) = -\sum_{k=1}^{\infty} \frac{z^{-k}}{\Gamma(\beta - \alpha k)} + \mathrm{O}(|z|^{-p-1}). \tag{A7}$$

APPENDIX B: ASYMPTOTICS OF THE FUNCTION $\mathcal{K}_{d,\sigma}(y)$

Here we will evaluate the asymptotic behaviors of the auxiliary function $\mathcal{K}_{d,\sigma}(y)$ used in the expression of the free energy and the quantities descending from it. It is defined by

$$\mathcal{K}_{d,\sigma}(y) = \frac{\sigma}{2(4\pi)^{d/2}} \int_0^\infty dx\, x^{-d/2-1} \left[\mathcal{A}\left(\frac{1}{4x}\right) - 1 \right] \times E_{\sigma/2,1}(-x^{\sigma/2} y), \tag{B1a}$$

where

$$\mathcal{A}(u) = \sum_{l=-\infty}^{\infty} e^{-ul^2}. \tag{B1b}$$

Using the identity

$$E_{\alpha,1}(-z) = 1 - zE_{\alpha,\alpha+1}(-z), \tag{B2}$$

it is possible to write down Eq. (B1a) in a more convenient form, which will allow us to extract the asymptotics of the function under investigation. After some algebra one obtains

$$\mathcal{K}_{d,\sigma}(y) = \sigma \pi^{-d/2} \Gamma\left(\frac{d}{2}\right) \zeta(d) - \frac{\sigma}{2} \mathcal{I}_{d,\sigma}(y), \tag{B3a}$$

where we have introduced the auxiliary function

$$\mathcal{I}_{d,\sigma}(y) = \frac{y}{(4\pi)^{d/2}} \int_0^\infty dx\, x^{\sigma/2-d/2-1} \left[\mathcal{A}\left(\frac{1}{4x}\right) - 1 \right] \times E_{\sigma/2,\sigma/2+1}(-x^{\sigma/2} y). \tag{B3b}$$

Now, setting $x = z(2\pi)^{-2}$ and with the help of the identity

$$\mathcal{A}(u) = \sqrt{\frac{\pi}{u}} \mathcal{A}\left(\frac{\pi^2}{u}\right), \tag{B4}$$

we rewrite Eq. (B3b) (after some algebra) in the form

$$\mathcal{I}_{d,\sigma}(y) = y \frac{\pi^{(d-1)/2}}{(2\pi)^\sigma} \int_0^\infty dx\, x^{\sigma/2-d/2-1/2} \left[\mathcal{A}(x) - \sqrt{\frac{\pi}{x}} - 1 \right] \times E_{\sigma/2,\sigma/2+1}\left(-y \frac{x^{\sigma/2}}{(2\pi)^\sigma} \right) + y \frac{\pi^{(d-1)/2}}{(2\pi)^\sigma} \times \int_0^\infty dx\, x^{\sigma/2-d/2-1/2} E_{\sigma/2,\sigma/2+1}\left(-y \frac{x^{\sigma/2}}{(2\pi)^\sigma} \right). \tag{B5}$$

The integral in the second term of the right-hand side of Eq. (B5) can be evaluated exactly with the help of the identities

$$u^{-\nu} = \frac{1}{\Gamma[\nu]} \int_0^\infty dt\, t^{\nu-1} e^{-ut} \tag{B6}$$

and

$$\int_0^\infty u^{\mu-1} \ln(a + bu^\nu) = \left(\frac{a}{b}\right)^{\mu/\nu} \frac{\pi}{\sin(\pi\mu/\nu)} \tag{B7}$$

to yield the result

$$2(d-1)^{-1} D_{d-1,\sigma} y^{(d-1)/\sigma}. \tag{B8}$$

For the evaluation of the first integral in the right hand side of Eq. (B5), we note that the two terms in the square brackets in Eq. (B5) cannot be integrated separately, since they diverge. Nevertheless, it is possible to outwit this divergence, by transforming further Eq. (B5) by adding and subtracting from the function $E_{\alpha,\alpha+1}(z)$ its asymptotic behavior at small arguments, leading, after some algebra, to

$$2\frac{y}{\sigma} \frac{\pi^{(d-1)/2}}{(2\pi)^\sigma} \frac{C_{d,\sigma}}{\Gamma[\sigma/2]} - 2d^{-1} D_{d,\sigma} y^{d/\sigma} + \mathcal{R}_{d,\sigma}(y). \tag{B9a}$$

Here we introduced the notations

$$C_{d,\sigma} = \int_0^\infty dx\, x^{\sigma/2-d/2-1/2} \left[2\sum_{l=1}^{\infty} e^{-xl^2} - \sqrt{\frac{\pi}{x}} \right], \quad d-1 < \sigma, \tag{B9b}$$

and

$$\mathcal{R}_{d,\sigma}(y) = 2y \frac{\pi^{(d-1)/2}}{(2\pi)^\sigma} \sum_{l=1}^{\infty} \int_0^\infty dx\, x^{\sigma/2-d/2-1/2} e^{-xl^2} \times \left[E_{\sigma/2,\sigma/2+1}\left(-y \frac{x^{\sigma/2}}{(2\pi)^\sigma} \right) - \frac{1}{\Gamma\left[\frac{\sigma}{2}+1\right]} \right]. \tag{B9c}$$

Collecting the above results, we obtain

$$\mathcal{K}_{d,\sigma}(y) = \sigma \pi^{-d/2} \Gamma\left(\frac{d}{2}\right) \zeta(d) - \sigma(d-1)^{-1} D_{d-1,\sigma} y^{(d-1)/\sigma} - y \frac{\pi^{(d-1)/2}}{(2\pi)^\sigma} \frac{C_{d,\sigma}}{T[\sigma/2]} + \sigma d^{-1} D_{d,\sigma} y^{d/\sigma} - \frac{\sigma}{2} \mathcal{R}_{d,\sigma}(y). \tag{B10}$$

The constant $C_{d,\sigma}$ introduced in Eq. (B9b) is the so-called Madelung constant (see, e.g. [62,63])

$$C_{d,\sigma} = \lim_{\delta \to 0} \left\{ 2\sum_{l=1}^{\infty} \frac{\Gamma[(\sigma-d+1)/2, \delta l^2]}{l^{(\sigma-d+1)/2}} - \int_{-\infty}^{\infty} dl \frac{\Gamma[(\sigma-d+1)/2, \delta l^2]}{l^{(\sigma-d+1)/2}} \right\}, \quad d-1 < \sigma, \tag{B11}$$

where $\Gamma[\alpha, x]$ is the incomplete gamma function. It has been shown that this constant has a remarkable property of symmetry [63], which relates its values in the case $d-1 < \sigma$ to those in the case $d-1 > \sigma$. On the other hand, it has been shown that $C_{d,\sigma}$ can be expressed in terms of the analytic continuation, over $d-1<\sigma$, of (for details see [63])

$$C_{d,\sigma} = 2\pi^{1/2+\sigma-d}\Gamma\left(\frac{d-\sigma}{2}\right)\zeta(d-\sigma), \quad d-1>\sigma. \tag{B12}$$

Equation (B10) is the general form of the functions $\mathcal{K}_{d,\sigma}(y)$. According to Eqs. (B11) and (B12) it can be used to investigate the critical behavior of the system for any dimension less than d.

For small y the asymptotic behavior of the function $\mathcal{K}_{d,\sigma}(y)$ is easily deduced from Eq. (B10). It is given by

$$\mathcal{K}_{d,\sigma}(y) \approx \begin{cases} \frac{\sigma}{\pi^{d/2}}\Gamma\left(\frac{d}{2}\right)\zeta(d) - \sigma\frac{|D_{d-1,\sigma}|}{d-1}y^{(d-1)/\sigma}, & 0<d-1<\sigma, \\ \frac{\sigma}{\pi^{d/2}}\Gamma\left(\frac{d}{2}\right)\zeta(d) - 2y[(4\pi)^{\sigma/2}\sigma\Gamma[\sigma/2]]^{-1}(1-\ln y), & \sigma=d-1, \\ \frac{\sigma}{\pi^{d/2}}\Gamma\left(\frac{d}{2}\right)\zeta(d) - y\frac{\pi^{(d-1)/2}}{(2\pi)^{\sigma}}\frac{C_{d,\sigma}}{\Gamma[\sigma/2]} + \sigma\frac{|D_{d-1,\sigma}|}{d-1}y^{(d-1)/\sigma}, & 0<\sigma<d-1. \end{cases} \tag{B13}$$

For large y the asymptotic behavior of the function $\mathcal{K}_{d,\sigma}(y)$ is obtained by substituting the large x behavior of the functions $E_{\alpha,\beta}(x)$ [given in Eq. (A7)] in the definition (B1a). After some calculations one ends up with

$$\mathcal{K}_{d,\sigma}(y) \simeq a_{d,\sigma}y^{-1}, \tag{B14a}$$

where

$$a_{d,\sigma} = \frac{2^{1+\sigma}}{\pi^{d/2}}\frac{\Gamma[(d+\sigma)/2]}{|\Gamma[-\sigma/2]|}\zeta(d+\sigma). \tag{B14b}$$

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